Exact Radiation Model For Perfect Fluid Under Maximum Entropy Principle

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We find an expression for mass of spherically symmetric system solving Euler-Lagrangian equation by Homotopy Perturbation Method. With the help of this expression and the Einstien Field equations we obtain an interior solutions set. Thereafter we explain different aspects of the solution describing the system in connection to mass, density, pressures, energy, stability, mass-radius ratio, compactness factor and surface redshift. This analysis shows that all the physical properties, in connection to brown dwarf stars, are valid with the observed features except that of stability of the model which seems suffers from instability.

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I. INTRODUCTION

There has always been search for the interior solution of spherically symmetric system. Considering different methods and approximations researchers tried to find out more than several hundreds different type of interior solutions. But very few solutions made its physical acceptance in all aspects describing the system so far.

However, in the present paper we have studied a spherically symmetric system of radiating star under the homotopy perturbation method (HPM). This is a series expansion method used in the solution of nonlinear partial differential equations, in the present case the Einstein field equations of general relativity. The method in principle employs a homotopy transform to generate a convergent series solution of differential equations. The HPM was introduced and developed by He [1–7] and others [8–12].

In one of his earlier works He [3] proposed a coupling method of a homotopy technique and a perturbation technique to solve non-linear problems. The author argued that in contrast to the traditional perturbation methods, the proposed method does not require a small parameter in the equation and the results reveal that the new method is very effective and simple. Later on several other researchers have employed the HPM in the diverse field of pure and applied mathematics (as an exhaustive appliance), and in physics and astrophysics (as a new field of application) to solve related non-linear differential equations in an extraordinary simplified way [13–17].

In connection to self-gravitating radiation model Sorkin et al. [18] have examined the entropy of self-gravitating radiation confined to a spherical box with finite radius in the context of general relativity. Their results are expected to supports the validity for self-gravitating systems of the Bekenstein upper limit on the entropy to energy ratio of material bodies.

Rahaman et al. [19] proposed and analyzed a model for the existence of strange stars where observed masses and radii are used to derive an interpolation formula for the mass as a function of the radial coordinate. The interpolation technique has been used to estimate the cubic polynomial that yield the following expression for the mass as a function of the radial coordinate $m(r) = ar^3 - br^2 + cr - d$ with a, b, c and d all being numerical constants. Their analysis is based on the MIT bag model and yields physically valid energy density, radial and transverse pressures.

However, in the above mentioned work of Rahaman et al. [19] the target object was a strange star. In our present investigation we start with the intention to de-

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velop a basic interior solution of the Einstein equations valid for any radiating model under a similar expression for the mass as a function of the radial coordinate $m(r)$. Then we match our theoretically obtained solutions set with the observational results for practical validity of the model and find that our model is best fit for the brown dwarf star of E0 type.

Therefore, the motivation of our present work is to find the interior solution of spherically symmetric system considering radiation effects in a more general way. For this we consider a metric with unknown time-time component. By using the maximum entropy principle (MEP) we obtain an Euler-Lagrangian equation which gives second-order differential equation for mass of the spherical distribution. Then we find out solution for mass by the homotopy perturbation method (HPM). The Einstien field equations are solved for density, time-time component of metric and pressures of the system. Thus, basically taking data set arbitrarily to see the physical nature and from there to predict whether this type of radiation model can effectively be used is the main purpose of the present investigation.

The work plan of the study is as follows: in Sec. II we develop the methodology for mass of the spherical symmetric system of radiating star under (i) the maximum entropy principle, and (ii) the homotopy perturbation method. Next in the Sec. III we provide the Einstein field equations for self-gravitating radiation system and discuss several physical aspects of the solution describing the system in connection to mass, density, pressures, energy, stability, mass-radius ratio, compactness factor and surface redshift. In Sec. IV it has been shown in details that all the physical features in connection to brown dwarf stars of type E0 are valid with the observed features except that of stability of the model which seems partially unstable. We have made some specific remarks on the radiation model under consideration in the Sec. V.

II. THE METHODOLOGY FOR MASS OF THE SPHERICAL SYMMETRIC SYSTEM OF RADIATING STAR

A. The Maximum Entropy Principle

We consider the metric of the spherical system

$$ds^2 = -g_{tt}(r)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1}$$

where $m(r)$ is the mass distribution of the spherically symmetric body and $g_{tt}(r)$ is the time-time component of metric. These are functions of radial coordinate r only.

The equation of state for the system with thermal radiation is given by

$$p_r = \frac{1}{3}\rho, \tag{2}$$

where, p_r and ρ are the radial pressure and density of the matter distribution. Compatible with spherically symmetry, we assume the general energy-momentum tensor as

$$T^{\mu}_{\nu} = (\rho + \frac{1}{3}\rho)u^{\mu}u_{\nu} - \frac{1}{3}\rho g^{\mu}_{\nu} + (p_t - \frac{1}{3}\rho)\eta^{\mu}\eta_{\nu}, \tag{3}$$

with $u^{\mu}u_{\mu} = -\eta^{\mu}\eta_{\mu} = 1$.

Here $u^{\mu}u_{\mu} = -\eta^{\mu}\eta_{\mu} = 1$ and $u^{\mu}\eta_{\nu} = 0$. The vector u^{μ} is the fluid 4-velocity of the local rest frame of the radiation and η^{μ} is the unit spacelike vector which is orthogonal to u^{μ}, where the radial pressure p_r is in the direction of η^{μ}.

For locally measured temperature T of black body radiation, the rest frame energy density and entropy density respectively assume the forms as [18]

$$\rho = bT^4, \tag{4}$$

$$s = \frac{4}{3}bT^3. \tag{5}$$

where b is a constant of order unity (in Planck units $G = c = h = k = 1$), on the assumption that the number of species of radiation is of order unity [20].

By substituting Eq. (4) into Eq. (5), one can easily obtain the following relation

$$s = \alpha(\rho)^{3/4}, \tag{6}$$

where $\alpha = \frac{4}{3}b^{1/4}$.

For the matter distribution up to radius $r \leq R$, the total entropy is given by

$$S = 4\pi \int_0^R s(r)\left[1 - \frac{2m(r)}{r}\right]^{-1} r^2 dr, \tag{7}$$

which can be simplified as

$$S = (4\pi)^{\frac{1}{4}}\alpha \int_0^R L dr, \tag{8}$$

where the Lagrangian L is given by

$$L = (m')^{\frac{3}{4}}\left[1 - \frac{2m(r)}{r}\right]^{-\frac{1}{2}} r^{\frac{1}{2}}, \tag{9}$$

with

$$m'(r) = 4\pi r^2 \rho, \tag{10}$$

which can be shown from the Einstein field equations $G_{ab} = 8\pi T_{ab}$ [18].

Also, we have Euler-Lagrangian equation in the form

$$\frac{d}{dr}\left(\frac{\partial L}{\partial m'}\right) - \frac{\partial L}{\partial m} = 0, \tag{11}$$

which reduces to

$$m' + m''m - \frac{1}{2}m''r - \frac{2}{3}(m')^2 - \frac{4mm'}{r} = 0. \tag{12}$$

This is the master equation for a self-gravitating radiation system. If it is possible to solve this highly nonlinear differential equation, then one would get complete structure of the self-gravitating radiation system and therefore could be used to describe a model of fluid sphere or stellar configuration. In the following section, we shall solve this equation by using a newly developed technique known as *homotopy perturbation method* (HPM) [3].

B. The Homotopy Perturbation Method

Primarily our task is to find the solution of Eq. (12) by using homotopy perturbation method. In this method we consider linear part of the differential equation as $m' = dm/dr$ and the rest of the terms as non-linear. So the homotopy equation can be written as [3]

$$m' - m_0' + h\left[m_0' + mm'' - \frac{1}{2}m''r - \frac{2}{3}(m')^2 - \frac{4mm'}{r}\right] = 0, \tag{13}$$

where h is a parameter and m_0 is the initial guess of the solution. For $h = 0$, we get initial approximation solution and for $h = 1$, one obtains the desired solution for mass.

We assume the initial solution as $m_0 = ar^3$. According to HPM, therefore, we consider the general solution as

$$m = m_0 + hm_1 + h^2m_2 + h^3m_3 + ... \tag{14}$$

Substituting this in Eq. (12) and equating coefficients of the different orders h, we get the solution for mass up to second order correction as follows

$$m(r) = ar^3 + \frac{36}{5}a^2r^5 + \frac{312}{35}a^3r^7 + 3C_1ar^2 + C_2, \tag{15}$$

where C_1 and C_2 are constants of integration.

The boundary conditions in the present system are

$$m(0) = 0, \quad m'(R) = 0, \tag{16}$$

with

$$m'(r) = 3ar^2 + 36a^2r^4 + \frac{312}{5}a^3r^6 + 6C_1ar. \tag{17}$$

So, we immediately obtain the constants of integration, C_1 and C_2 in the following form

$$C_1 = -\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right), \tag{18}$$

and

$$C_2 = 0, \tag{19}$$

where R is the radius of the spherical distribution.

Therefore, substituting C_1 in Eq. (15), we obtain

$$m(r) = ar^3 + \frac{36}{5}a^2r^5 + \frac{312}{35}a^3r^7 -3ar^2\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right). \tag{20}$$

III. THE EINSTEIN FIELD EQUATIONS FOR SELF-GRAVITATING RADIATION SYSTEM

For the matter distribution given in Eq. (2), the Einstein field equations for the metric (1) can be written as

$$\frac{2m'}{r^2} = 8\pi\rho, \tag{21}$$

$$\frac{2m}{r^3} - \left(1 - \frac{2m}{r}\right)\frac{g_{tt}'}{g_{tt}}\frac{1}{r} = -\frac{8\pi}{3}\rho, \tag{22}$$

$$-\left(1 - \frac{2m}{r}\right)\left[\frac{1}{2}\frac{g_{tt}''}{g_{tt}} - \frac{1}{4}\left(\frac{g_{tt}'}{g_{tt}}\right)^2 + \frac{1}{2r}\frac{g_{tt}'}{g_{tt}}\right] - \left(\frac{m}{r^2} - \frac{m'}{r}\right)\left(\frac{1}{r} + \frac{1}{2}\frac{g_{tt}'}{g_{tt}}\right) = -8\pi p_t. \tag{23}$$

From Eq. (21) we get the density as

$$\rho = \frac{m'(r)}{4\pi r^2}. \tag{24}$$

Therefore, substituting Eq. (17) into Eq. (24) we obtain an expression for density as follows

$$\rho = \frac{1}{4\pi}\left[3a + 36a^2r^2 + \frac{312}{5}a^3r^4\right] - \frac{3a}{2\pi r}\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right). \tag{25}$$

Again by using Eq. (2) and Eq. (24), we solve Eq. (22) for g_{tt} as follows:

$$g_{tt} = K\frac{\exp\int\frac{4}{3(r-2m)}}{r(r-2m)^{1/3}}, \tag{26}$$

where K is a constant.

It can be noted that for negative values of a and any value of r the exponential term becomes unity and hence we obtain

$$g_{tt} \approx \frac{K}{r(r-2m)^{1/3}}. \tag{27}$$

Using boundary conditions for g_{tt}, one can find out K, which ultimately gives the time-time component of metric (1) as

$$g_{tt} = \left(\frac{R}{r}\right)^{4/3}\frac{(1 - \frac{2m(R)}{R})^{4/3}}{(1 - \frac{2m(r)}{r})^{1/3}}, \tag{28}$$

where $m(R)$ is a constant and can be given by

$$m(R) = -\frac{1}{2}aR^3 - \frac{54}{5}a^2R^5 - \frac{156}{7}a^3R^7. \tag{29}$$

IV. PHYSICAL FEATURES OF THE RADIATION MODEL

A. Density and Mass

To start with the basic physical studies preferably we choose parameter $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius and $m(R) = 0.080$ solar mass of $E0$ class of brown dwarf stars (see the Link [22]). The reason to consider these data set lies on the physical background that our target is to investigate a radiating model which is compatible with brown dwarfs which are cool star-like objects that have insufficient mass to maintain stable nuclear fusion in their core regions [23]. It has been argued [23] that although brown dwarfs are not stars, they are expected to form in the same way, and they should radiate a large fraction of their gravitational energy at near-infrared wavelengths.

For the above specification we plot variation of mass and density as shown in Fig. 1 and 2.

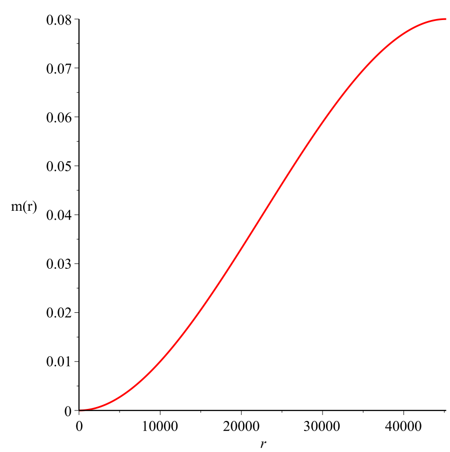

FIG. 1: Variation of mass as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

B. Pressure and anisotropy

Using Eq. (2) and (25), we get an expression for radial pressure as

$$p_r = \frac{1}{12\pi}\left[3a + 36a^2r^2 + \frac{312}{5}a^3r^4 - \frac{6a}{r}\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right)\right], \quad (30)$$

whereas from Eq. (23) we get tangential pressure as follows

$$p_t = \frac{1}{72\pi r^2}\left[3m''r - 10m' + \frac{2m}{r} + 4\frac{(m' - \frac{m}{r})^2}{(1 - \frac{2m}{r})} + 4\right]. \quad (31)$$

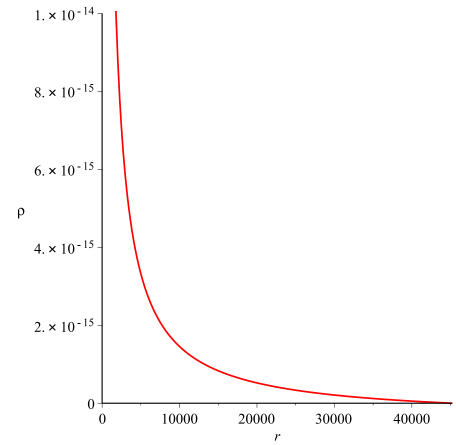

FIG. 2: Variation of density as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

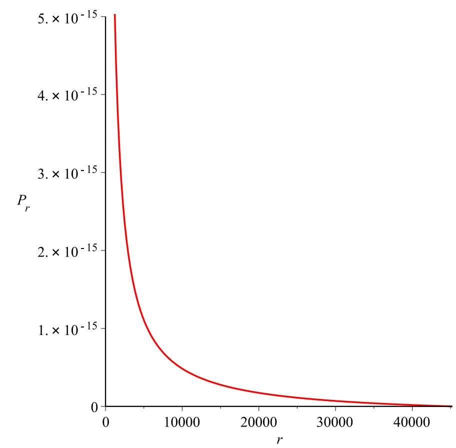

FIG. 3: Variation of radial pressure as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

From Figs. 3 and 4 we observe that the radial and tangential pressures and hence anisotropy (Fig. 5) are decreasing function of radial distance r.

C. Stability

In connection to stability analysis we have checked out the stability criteria given by Herrera [21]. In this approach (usually known as the concept of cracking or overturning) the region for which the radial and tangential sound speeds $v_{st}^2 - v_{sr}^2 < 0$ and its numerical value less than unity is a stable region. We observe from the plot in Fig. 6 that for $v_{st}^2 - v_{sr}^2$ there is no change of sign but up to 41800 km its numerical value greater than unity. This implies that our configuration with E0 type brown dwarf star of 0.065 Solar radius (i.e. 45227 km) is unfortunately suffers from instability within the region of core radius 41800 km.

However, it is interesting to observe from the Fig. 7

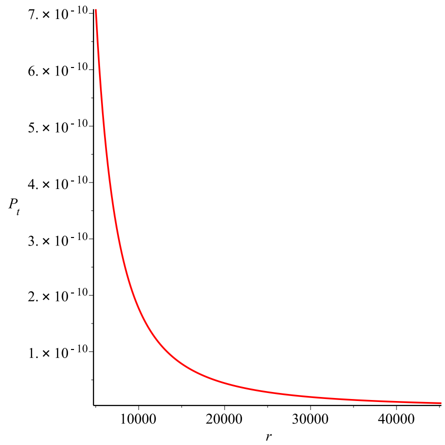

FIG. 4: Variation of tangential pressure as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

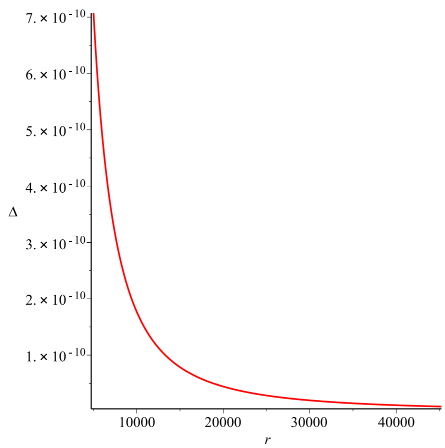

FIG. 5: Anisotropy, $\Delta = (p_t - p_r)$, of the pressures as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

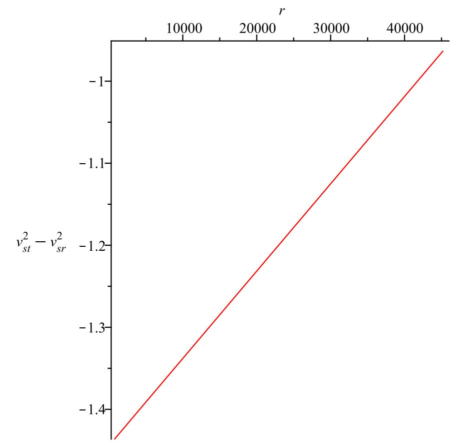

FIG. 6: Sound speed $v_{st}^2 - v_{sr}^2 < 0$ through out the region of core radius (41800 km) implying instability of configuration with E0 type brown dwarf star of 45227 km where unit of r is in km

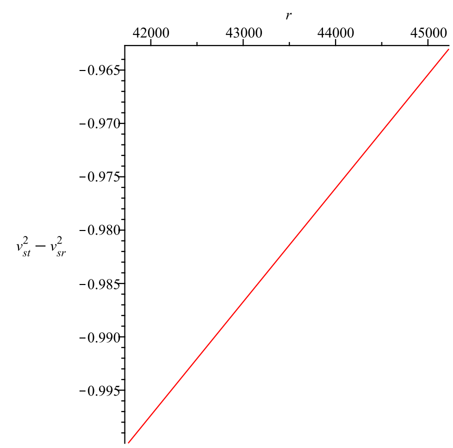

FIG. 7: Sound speed $v_{st}^2 - v_{sr}^2 < 0$ through out the region of the thin shell (3427 km) implying stability of configuration with E0 type brown dwarf star of 45227 km where unit of r is in km

that outside the core our model is potentially stable one where clearly $v_{st}^2 - v_{sr}^2 < 0$ and its numerical value less than unity. Therefore, from the plot of Fig. 7 it is obvious that within the shell starting approximately from 41800 km to 45227 km the value of sound speed differences is less than one which is the condition of stability. Like wise the plot is maintaining same sign so there is no crack which is also another condition of stability. Hence only 7.57% of the shell (3427 km) near the surface the Brown Dwarf has a stable configuration.

D. Energy conditions

Let us now check whether all the energy conditions regarding present radiating model are satisfied or not. For this purpose, we are considering the following inequalities and plotting graphs for each case:

$$(i)\ NEC : \rho + p_r \geq 0,\ \ \rho + p_t \geq 0,$$

$$(ii)\ WEC : \rho + p_r \geq 0,\ \ \rho \geq 0,\ \ \rho + p_t \geq 0,$$

$$(iii)\ SEC : \rho + p_r \geq 0,\ \ \rho + p_r + 2p_t \geq 0.$$

We see from Figs. 8 and 9 related to different energy conditions indicate that in our model all the energy conditions are satisfied through out the interior region.

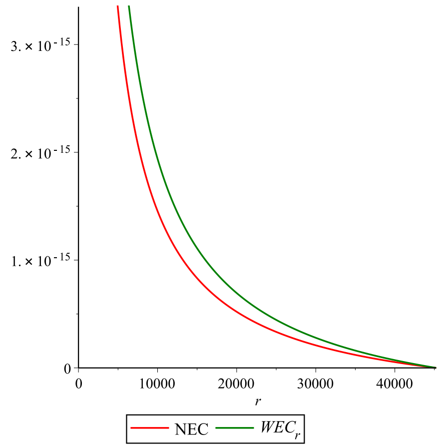

FIG. 8: The energy conditions applicable in the interior of the spherical distribution have been plotted against r for NEC and WEC_r

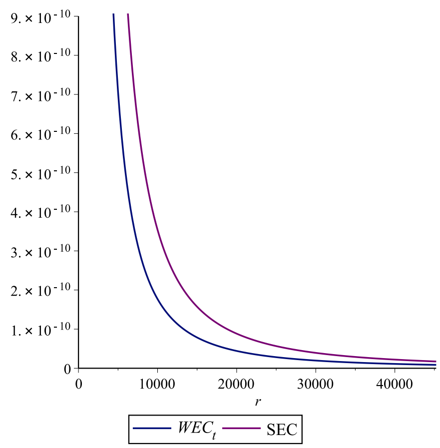

FIG. 9: The energy conditions applicable in the interior of the spherical distribution have been plotted against r for SEC and WEC_t

E. Compactness and Redshift

By definition compactness of a star can be given by $u = m(r)/r$ which in our case takes the following form:

$$u = ar^2 + \frac{36}{5}a^2r^4 + \frac{312}{35}a^3r^6 - 3ar\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right). \tag{32}$$

Fig. 10 shows that $\frac{m(R)}{R} < \frac{4}{9}$.

Again, surface redshift is defined by the relation

$$z = (1 - 2u)^{-\frac{1}{2}} - 1, \tag{33}$$

which in the present study turns out to be

$$z = \frac{1}{\sqrt{[1 - 2\{ar^2 + \frac{36}{5}a^2r^4 + \frac{312}{35}a^3r^6 - 3ar\left(\frac{R}{2} + 6aR^3 + \frac{52}{5}a^2R^5\right)\}]}} - 1. \tag{34}$$

The variation of surface redshift z is shown in Fig. 11 which is physically within an acceptable profile.

V. CONCLUSION

In the present paper we have searched for an expression of mass with spherically symmetric system to solve Euler-Lagrangian equation by Homotopy Perturbation Method. By employing this expression for mass and the Einstien Field equations we have obtained an interior solution. We have also explained different physical properties of the solution describing the system. However, in the present model for radiating brown dwarf stars all the features are seen to be physically viable except that of stability of the matter distribution.

In connection to these aspects with brown dwarf stars we have done a preliminary calculation taking the data set of Bhar et al. [24] related to highly compact stars such as neutron stars and strange stars. This data set

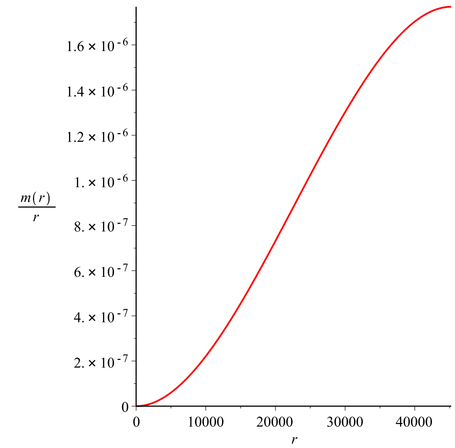

FIG. 10: Variation of compactness as a function of radial coordinate $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

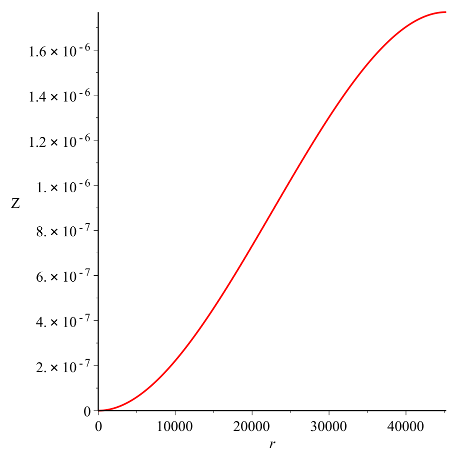

FIG. 11: Variation of redshift z as a function of radial distance r for $a = -1.73 \times 10^{-15}$ and $R = 0.065$ solar radius

provides negative radial pressure and also in this case the model suffers from instability.

Therefore, we suspect that -

(1) radiation model may not be compatible with highly compact stars rather it is compatible with radiating brown dwarf stars. Actually, to relate radiation model with highly compact stars is also not justified due to their exhausted fuel system and hence burning as well as radiation stage which seems have been stopped much earlier.

(2) instability may be inherent property of any radiating compact stars under HPM and MEP as observed in the present model with brown dwarf stars. Instability has also been observed in the extension of the work of Bhar et al. [24] related to highly compact stars such as neutron stars and strange stars.

As a final remark we would like to state that some more investigations are needed to perform with different methodology before coming to a definite decision regarding (i) applicability of HPM and MEP to radiating and highly compact stars, and (ii) instability of radiating brown dwarf stars under HPM and MEP.

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